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The spectrum of closed loops of fundamental flux in D=2+1 $SU(N)$ gauge theories.

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We study the closed-string spectrum of SU(N) gauge theories in the fundamental representation in 2+1 dimensions. We calculate the energies of the lowest lying ~ 30 states using a large variety of operators characterised by the quantum numbers of parity and longitudinal momentum. We find that our results for the ground state are very well approximated by the Nambu-Goto (NG) predictions even for short strings. For the excited states, we observe significant deviations from the NG predictions only for very short strings and they decrease rapidly with increasing string length. Finally, we see that Nambu-Goto provides a much better description of our results than the effective string theoretical predictions. We discuss the continuum and large-N limits.

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1. Introduction

It is an old idea that large-N QCD might be exactly reformulated as a string theory. Although this relation has never been made precise, the results obtained in lattice gauge theory leave little doubt that there is a connection between these two theories. One of these results that leads to such a conclusion is the energy spectrum of the flux-tube. A flux-tube in $SU(N)$ gauge theories with length l much larger than its width is expected to be described by an effective low energy string theory. This can be verified by studying the energy spectrum of the flux-tube and comparing it to the effective string theory prediction.

In this work we focus on the spectrum of the closed fundamental flux-tube in pure $SU(N)$ gauge theories in $D = 2+1$ dimensions. We calculate the energies of the lowest lying $\sim$30 states. This allows to extend the comparison with theoretical predictions to states with more quantum numbers and a nontrivial degeneracy structure. More precisely, we want to check whether the closed flux-tube can be described by the Nambu-Goto (NG) string model [1], and, if so, how good a description it is.

The flux-tubes that we study have lengths that range from ~ 0.65 fm to ~ 2.60 fm and the lattices we use have spacings $a \simeq 0.04$ fm and $a \simeq 0.08$ fm (Despite working in pure gauge theories and in $D = 2+1$, we define 1fm through the convention $\sigma \equiv (440\mathrm{MeV})^2$). The gauge groups that we study have $N = 3,6$. The use of $N > 3$ will suppress any mixing amplitudes (eg. glueball-string mixing) that cannot be described by a simple low-energy effective string theory model.

The study of confining flux-tubes with lattice techniques has been an active field for the past three decades, and we refer the reader to some recent papers [2]. For more details on the calculation and relevant references see our longer write-up [3].

2. Lattice construction

We define our gauge theory on a three-dimensional periodic Euclidean space-time lattice with $L \times L_\perp \times L_T$ sites. It is important to mention that in order to minimize the finite volume effects, we increase $L_\perp$ and L_T as we decrease the length of the flux-tube. For the calculation of the physical observables we perform Monte-Carlo simulations using the standard Wilson plaquette action:

$$S_{\mathrm{W}} = \beta \sum_P \left[1 - \frac{1}{N} \mathrm{ReTr} U_P \right], \tag{2.1}$$

The bare coupling β is related to the dimensionful coupling g^2 through $\lim_{a\to 0} \beta = 2N/ag^2$. In the large–$N$ limit, the 't Hooft coupling $\lambda = g^2 N$ is kept fixed, and so we must scale $\beta = 2N^2/\lambda \propto N^2$ in order to keep the lattice spacing fixed. The simulation we use mixes standard heat-bath and over-relaxation steps in the ratio 1 : 4. These are implemented by updating $SU(2)$ subgroups using the Cabibbo-Marinari algorithm.

We have calculated the string spectrum for the case of $SU(3)$ with $a \simeq 0.04, 0.08$ fm and $SU(6)$ with $a \simeq 0.08$ fm. For the case of $SU(3)$ and $a \simeq 0.08$ fm, the string lengths ranged between ~ 0.65fm and ~ 2.60fm and for the two other cases between ~ 0.95fm and ~ 2.0fm.

3. General strategy

We calculate the energies of flux-tubes that are closed around a spatial torus using the correlators of suitably smeared Polyakov loops that wind once around the corresponding spatial tori and that have vanishing transverse momentum. This is a standard technique with smearing/blocking designed to enhance the projection of our operators onto the physical states. We classify our operators using the quantum numbers of transverse parity in $P = \pm$ and winding momentum $q = 0, \pm 1, \pm 2, \ldots$ in units of $2\pi/l$.

For each combination of these quantum numbers we construct the full correlation matrix of operators and use it to obtain best estimates for the string states using a variational method applied to the transfer matrix $\hat{T} = e^{-aH}$.

Since we are mostly interested in the excited states, it makes sense to introduce transverse deformations in the simple Polyakov loop. Using this procedure we can increase the number of different operators to 80-200 and construct Polyakov loops described by the quantum numbers of P and q. We present the different paths used in the calculation in Table 1.

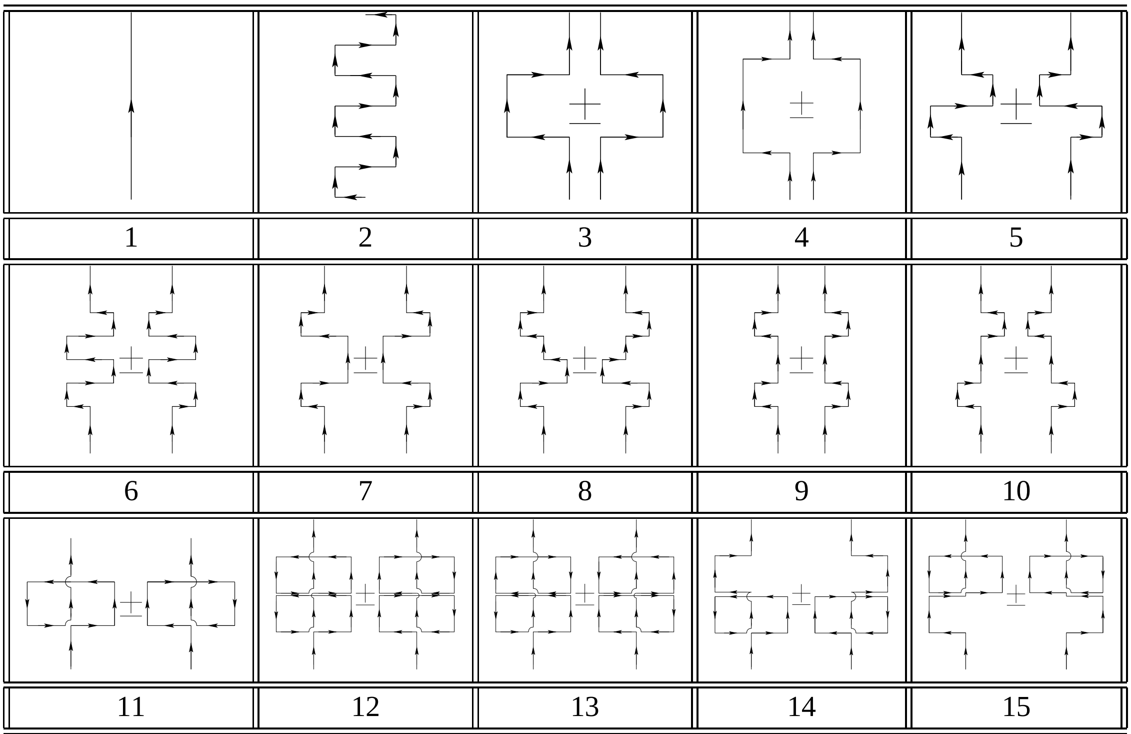

Table 1: The lattice paths used in the construction of Polyakov loops in this work. Our set of operators can be divided into three subsets: a. The simple line operator (1) in several smearing/blocking levels, b. The wave-like operator (2) whose number depends upon L, $L_\perp$, and the smearing/blocking level, c. The pulse-like operators (3-15) in several different smearing/blocking levels.

4. Theoretical expectations: The spectrum of the Nambu-Goto String model

The action of the Nambu-Goto (NG) model [1] is the area of the worldsheet swept by the propagation of the string. This model is quantum-mechanically inconsistent due to the Weyl anomaly ($D \neq 26$), but since this anomaly is suppressed for long strings [4] it can still be considered as an effective low energy model. The classical ground state of the model is the worldsheet configuration with the minimal area, and the transverse area fluctuation around this ground state constitute the quantum spectrum of the model. These fluctuations are usually referred to as left/right movers, depending on the momentum that they carry in the direction parallel to the string axis.

The single string states can be characterised by the winding number w which indicates how many times the string winds around the torus, by the occupation number $n_{L,(R)}(k)$ and the energy k of the left and right movers and also by the center of mass momentum $\vec{p}_{\mathrm{c.m.}}$. By projecting to zero transverse momentum we are left only with the momentum along the string axis which is quantized in units of $2\pi q/l$ with $q = 0, \pm 1, \pm 2, \ldots$. These quanta are not independent of $n_{L,R}$ and obey the level matching constraint $N_L - N_R = qw$, where $N_{L(R)}$ enumerates the momentum contribution of the left(right) movers in a certain state as follows:

$$N_L = \sum_{k>0} \sum_{n_L(k)>0} n_L(k)k \quad \text{and} \quad N_R = \sum_{k'>0} \sum_{n_R(k')>0} n_R(k')k' \tag{4.1}$$

The string states can be characterised as irreducible representations of the $SO(D-2)$ group that rotates the spatial directions transverse to the string axis. In our $D = 2+1$ case this group becomes the transverse parity with eigenvalues $P = (-1)^{\sum_{i=1} n_L(k_i) + \sum_{j=1} n_R(k'_j)}$. Finally, the energy of a closed-string state described by the above quantum numbers for any D is given by the following relation:

$$E^2_{N_L,N_R,q,w} = (\sigma l w)^2 + 8\pi\sigma \left(\frac{N_L + N_R}{2} - \frac{D-2}{24} \right) + \left(\frac{2\pi q}{l} \right)^2 . \tag{4.2}$$

5. Theoretical expectations: Effective string theories

Since in 2+1 dimensions the NG string is at best an effective low-energy string theory, it makes sense to generalise it and write the most general form of an effective string action consistent with the symmetries of the flux-tube system. This was first done in the early eighties for the case of $w = 1$ and $q = 0$ in [5], and the spectrum obtained for any D was given by:

$$E_n = \sigma l + \frac{4\pi}{l} \left(n - \frac{D-2}{24} \right) + \mathscr{O}(1/l^2), \tag{5.1}$$

where $n = 0, 1, 2, \ldots$. The second term in the above formula is known as the Lüscher term and is expected to be universal. One can easily show that the NG model obeys this universality by expanding the square-root of Eq. (4.2) to leading order in $1/l$.

Recently, the work [5] was extended in [6], where the authors used an open-closed string duality of the effective string theory. Using this duality they showed that for any D the $\mathscr{O}(1/l^2)$ is absent from Eq. (5.1), and that in $D = 2+1$ the $\mathscr{O}(1/l^3)$ has a universal coefficient. Consequently, in $2+1$ dimensions Eq. (5.1) is extended to:

$$E_n = \sigma l + \frac{4\pi}{l} \left(n - \frac{1}{24} \right) - \frac{8\pi^2}{\sigma l^3} \left(n - \frac{1}{24} \right)^2 + \mathscr{O}(1/l^4), \tag{5.2}$$

or in the equivalent form:

$$E_n^2 = (\sigma l)^2 + 8\pi\sigma \left(n - \frac{1}{24} \right) + \mathscr{O}(1/l^3). \tag{5.3}$$

The form Eq. (5.3) is particularly convenient since the two first terms on the right hand side are the predictions of the NG model that we find to be a very good approximation.

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Motivated by this recent development, we decided to fit our data using the following ansatz:

$$E_{\text{fit}}^2 = E_{NG}^2 - \sigma \frac{C_p}{(l\sqrt{\sigma})^p}, \qquad p \geq 3 \tag{5.4}$$

where E_{NG}^2 is the Nambu Goto prediction given by Eq. (4.2) for $w = 1, n = N_L = N_R$ and where C_p are dimensionless coefficients that in general can depend on the quantum numbers of the state.

6. Results

In this section we present our results from the calculation. The results were obtained for $SU(3)$ with $\beta = 21.00, 40.00$ and for $SU(6)$ with $\beta = 90.00$. We have focused on strings with $1.4 \lesssim l\sqrt{\sigma} \lesssim 5.5$. All energies that we present were obtained from single cosh fits to the correlation functions of our 'best' operators - see discussion in Section 3.

We present the results in Figs. 1,2. The lines are the predictions of the NG model. The string tensions used for these predictions were extracted from the ground state energies of the $q = 0$ calculations with the use of the ansatz Eq. (5.4) for $p = 3$. The fitting parameters are the dimenionless C_3 and the lattice spacing in physical units $a\sqrt{\sigma}$. We present these in Table 2.

Gauge group	β	$a^2\sigma$	C_3	Confidence level
$SU(3)$	$\beta = 21.00$	0.030258(26)	0.160(21)	69%
	$\beta = 40.00$	0.007577(13)	0.05(31)	15%
$SU(6)$	$\beta = 90.00$	0.029559(36)	0.04(21)	88%

Table 2: The parameters $a^2\sigma$ and C_3 in the fit.

According to our results the NG predictions are very good approximations to the flux-tube spectrum and deviate from our data only at the level of $\sim 2\%$ when $l\sqrt{\sigma} \gtrsim 4.2$. In contrast to that, a comparison of our data to Eq. (5.1) and/or Eq.(5.2) fails even when $l\sqrt{\sigma} \gtrsim 4$. This is an important point that tells us that the higher $O(1/l^4)$ terms in Eq. (5.2) are significant for the excited states at these lengths, and that they are captured quite well by Eq. (4.2) (to the level of $\sim 2\%$, a deviation that may or may not be due to a percent level systematic errors that we did not aim to control for the excited states). Finally, it is important to remark that the degeneracy pattern predicted by the NG model is seen from our data. For example, the second energy level is fourfold degenerate at large-l. This degeneracy includes two positive parity states and two negative parity states, and these start splitting significantly once $l\sqrt{\sigma} \lesssim 3$.

Next, we performed fits of the energy of the excited states. In the case of the first excited energy level, where there is only one state per level, we used the fitting ansatz Eq. (5.4). In the case of the second excited energy level, where for each parity there are two states, we fitted the difference between the energies squared of these states. The fits showed that to unambiguously determine the power p, and test the Lüscher-Weisz prediction of [6], we need statistical errors which are at least 2-3 times smaller than the ones our data has, and a simultaneous control of any systematic errors that may be important at the level of a few percents accuracy.

Finally, we move to the non-zero longitudinal momentum $q \neq 0$ calculation. In Section 4 we have mentioned that the number of left and right movers in the NG prediction is constrained by the

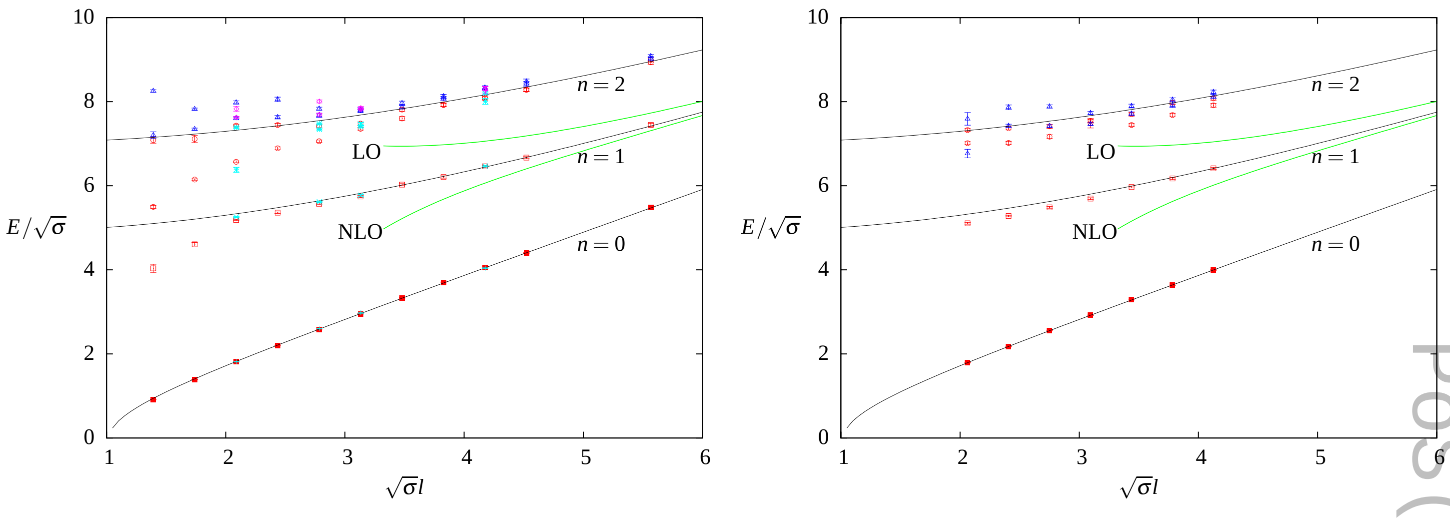

Figure 1: The energies of the first three energy levels divided by $\sqrt{\sigma}$ as a function of $l\sqrt{\sigma}$. The three lines are the NG predictions for the ground state and the excited states. The green lines are the NG predictions expanded to leading order Eq. (5.1) and next-to-leading order Eq. (5.2) for $n=1$. Left panel: We present the results for the case of $SU(3)$ for two different values of β (two different lattice spacings). For $\beta=21$, positive parity states are presented in red and negative parity states in blue. For $\beta=40$, positive parity states are presented in cyan and negative parity states in pink. Right panel: We present the results for the case of $SU(6)$ with $\beta=90$. Positive parity states are presented in red and negative parity states in blue.

level matching condition $N_L - N_R = qw$. The comparison of this prediction to our data is presented in Fig. 2, where we plot $\sqrt{E^2/\sigma - \left(2\pi q/\sqrt{\sigma}l\right)^2}$ as a function of $l\sqrt{\sigma}$. It is clearly seen that our data is very well described by Eq. (4.2).

7. Summary

We have calculated the energy spectrum of closed strings in the fundamental representation of $SU(N)$ gauge theories in 2+1 dimensions. To perform this calculation we constructed a basis of $\sim 80-200$ operators for each configuration of quantum numbers. The use of this large basis of operators is convenient for two reasons. Firstly, it is possible to extract masses of excited states with high confidence since it increases the overlaps of our lattice operators onto the physical states (compared to what we observe using the simple line operator in 5 smearing/blocking levels). Secondly, it enables us to study states with quantum numbers like the transverse parity P and the longitudinal momentum q.

Comparing our results to different theoretical predictions, we find that the agreement with the NG prediction in Eq. (4.2) is very good, including the expected degeneracy pattern. This agreement is in striking contrast to what we find when we simply compare to the Lüscher term as in Eq. (5.1) or to the Lüscher-Weisz prediction in Eq. (5.2). While for the ground states these describe our data already at $l\sqrt{\sigma}\simeq 3$ where they are indistinguishable from the full NG formula, for the excited states the situation is completely different. In particular, whereas the NG prediction works well for the first excited state already at $l\sqrt{\sigma}\simeq 3$, the predictions of Eq. (5.1) and Eq. (5.2) are still very far

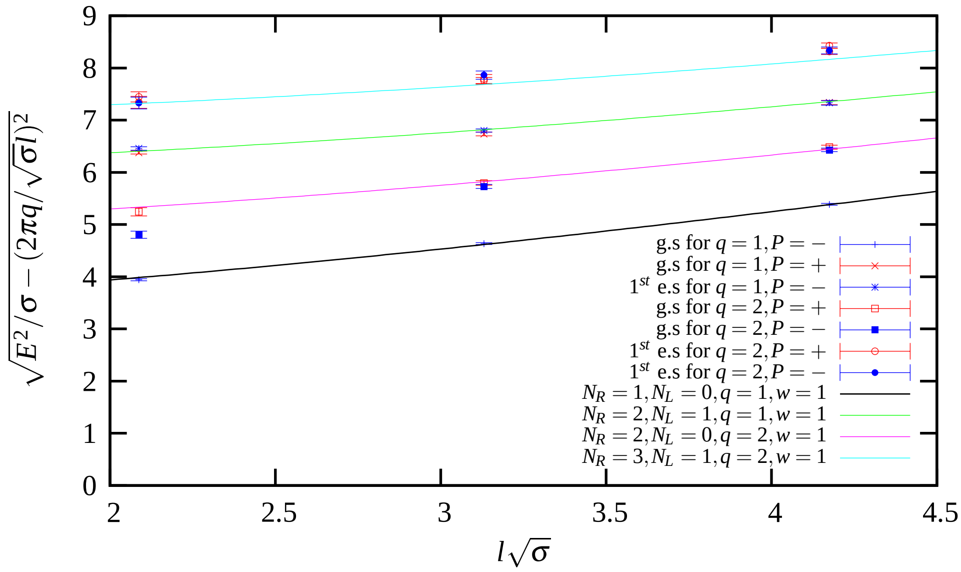

Figure 2: $\sqrt{E^2/\sigma - (2\pi q/\sqrt{\sigma} l)^2}$ as a function of $l\sqrt{\sigma}$ for the ground state and excited states of the non-zero momentum along the string direction. The four lines present the NG predictions Eq. (4.2). For $N_R = 3, N_L = 1$ and $q = 2$, the expected degeneracy is three, which agrees with our excited state data, i.e. two(one) states with positive(negative) parity in red(blue).

from the data (see Figure. 1). This demonstrates that a confining flux-tube can be described by a covariant string, with small or moderate corrections down to very short lengths.

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